Topology and its Applications 201 (2016) 350–357

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Polynomial invariants of a link with local symmetry

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ARTICLE INFO

Article history:
Received 29 December 2014
Accepted 16 April 2015
Available online 21 December 2015

MSC:
57M25
57M27

Keywords:
Kauffman bracket polynomial
Adequate knot
Local symmetry

ABSTRACT

Let D be a link diagram and T a 4-tangle. By replacing each crossing of D by T, we get a new diagram $D \otimes T$, called a link diagram with local symmetry or tensor product of D and T. In this paper, we will study polynomial invariants of the link diagram $D \otimes T$ with local symmetry in terms of D and T, and as an application, we will study the adequacy of $D \otimes T$.

1. Introduction

Let D be a diagram of a link L. A *state* s of D is an assignment of characters A and B to each crossing of D as Fig. 1. Given a state s of D, we can splice each vertex by giving either A-*split* or B-*split*, see Fig. 1. Clearly, there are $2^{c(D)}$ different states where $c(D)$ is the number of crossings in D.

For a given state s of D, let $a(s)$ and $b(s)$ denote the numbers of the character A and the character B which are assigned to crossings of D respectively, and let $|s|$ denote the number of loops of the resulting diagram obtained from D by splicing all crossings of D according to its state. The *Kauffman bracket polynomial* $< D >\in \mathbb{Z}[A, B, d]$ of the link diagram D is given by

$$< D >= \sum_s A^{a(s)} B^{b(s)} d^{|s|-1}.$$

It is well-known that the Kauffman bracket polynomial $< D >$ is a regular isotopy invariant if $B = A^{-1}$ and $d = -A^2 - A^{-2}$, and that

$$V_L(A) = (-A)^{-w(D)} < D >$$

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http://dx.doi.org/10.1016/j.topol.2015.12.046
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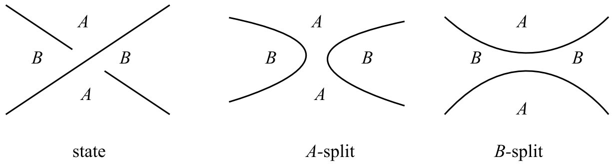


Fig. 1. A-split and B-split.

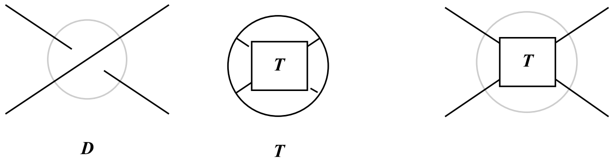


Fig. 2. $D \otimes T$.

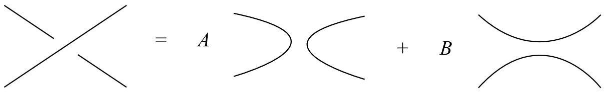
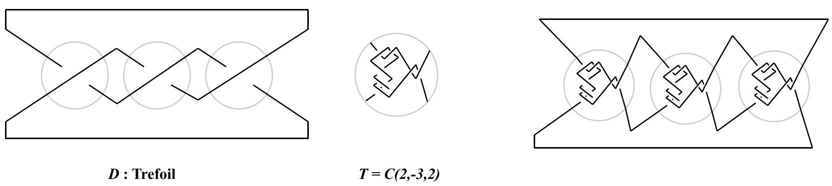


Fig. 3. The skein relation.

is the *Jones polynomial* of L, which is an isotopy invariant of L. Here $w(D)$ denotes the *writhe* of D.

Let D be a link diagram and T a 4-tangle. By replacing each crossing of D by T as in Fig. 2, we get a new diagram $D \otimes T$, called a *link diagram with local symmetry* or *tensor product of* D *and* T.

Example 1. Let D be the following diagram of the trefoil knot and let T be the diagram of the rational tangle $C(2, -3, 2)$. Then tensor product $D \otimes T$ of D and T is the right diagram of the following figure.



In this paper, we will study the Kauffman bracket polynomial $< D \otimes T >$ and the adequacy of the link diagram $D \otimes T$ with local symmetry.

2. Kauffman bracket polynomial of a link diagram with local symmetry

The bracket polynomial can be characterized by the skein module. The *skein module* $E(A, B, d)$ is the complex vector space generated by all link diagrams with the following relations:

(i) ambient isotopy in the plane;
(ii) $D \cup \bigcirc = d \cdot D$, where D is an arbitrary link diagram and $\bigcirc$ is a simple closed curve bounding a disk in the complement of D;
(iii) The skein relation for Kauffman bracket polynomial (see Fig. 3).

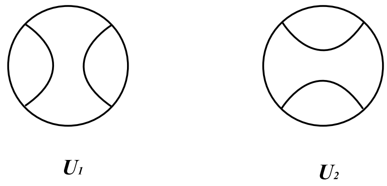


Fig. 4. Generators of $E_4(A, B, d)$.

From the relations (ii) and (iii), we know that $E(A, B, d)$ is an 1-dimensional vector space generated by the empty diagram. Each link diagram D represents an elements $[D]$, called the *skein class* of D, in $E(A, B, d)$. The Kauffman bracket polynomial is

$$< D >= d^{-1}[D].$$

Let $E_4(A, B, d)$ be the complex vector space generated by all 4-tangles quotiented by relations (i), (ii) and (iii) in the definition of $E(A, B, d)$. The skein module $E_4(A, B, d)$ is generated by two 4-tangles U_1 and U_2 in Fig. 4.

Since the diagram $D \otimes T$ is obtained by replacing each crossing of D by T as in Fig. 2, $D \otimes U_1 = s_A(D)$ and $D \otimes U_2 = s_B(D)$, where $s_A = s_A(D)$ and $s_B = s_B(D)$ are the extreme states which assign A and B to all crossings of D, respectively.

Theorem 2. *Let D be a diagram of a link and T a 4-tangle. Let $< D > (A, B, d)$ denote the Kauffman bracket polynomial of D. Suppose that $T = f(A)U_1 + g(A)U_2$ in $E_4(A) = E_4(A, B, d), B = A^{-1}, d = -A^2 - A^{-2}$. Then*

$$< D \otimes T >=< D > (f(A), g(A), -A^2 - A^{-2}).$$

Proof. We will start the proof by recalling the skein resolution tree for Kauffman bracket polynomial. To calculate $< D >$, we first choose any crossing c of D, which is the upmost crossing in the skein resolution tree of D, and then, apply the skein relation to c to obtain two diagrams at the next level in the skein resolution tree of D. At each diagram D' of the next level, choose any crossing c' of D' and apply the skein relation to c' to obtain four diagrams at the next level in the skein resolution tree of D. By repeating this steps, one can have a skein tree with $2^{c(D)}$ branches. At the end of each branch of the skein tree, the resulting diagram, called the *branch-end* diagram, has no crossings For a branch-end diagram s, let $a(s)$ and $b(s)$ denote the numbers of A's and B's which are assigned on the corresponding branch, respectively. Then

$$< D >= \sum_{s:\text{branch-end}} < D; s >$$

where $< D; s >= A^{a(s)}B^{b(s)}d^{|s|-1}$ for each branch-end diagram s.

Now consider the calculation of the Kauffman bracket polynomial $< D \otimes T >$. One can construct a skein resolution tree for the tangle T by the same method for D. Notice that the resulting branch-end diagram of the skein tree for T consists of either U_1 or U_2 in Fig. 4 and a number of disjoint circles. If we replace each disjoint circle with $-A^2 - A^{-2}$, then the branch-end diagram of the resulting skein tree is either U_1 or U_2. Indeed, $f(A)$ and $g(A)$ in the expression $T = f(A)U_1 + g(A)U_2$ are obtained by this process when B is replaced with A^{-1}.

Now, choose the tangle T in $D \otimes T$ whose corresponding crossing in D is c, the upmost crossing in the skein resolution tree of D. By applying the skein relation to all crossing of T to get a skein resolution tree of T, one can obtain the top level of the skein resolution tree of $D \otimes T$ at the right of the Fig. 5. By the

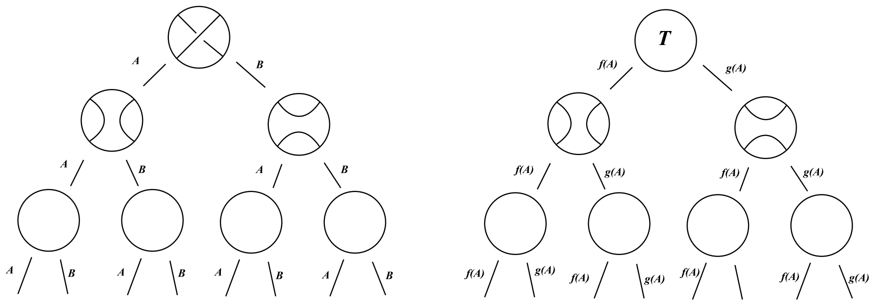


Fig. 5. The skein resolution trees of D and $D \otimes T$.

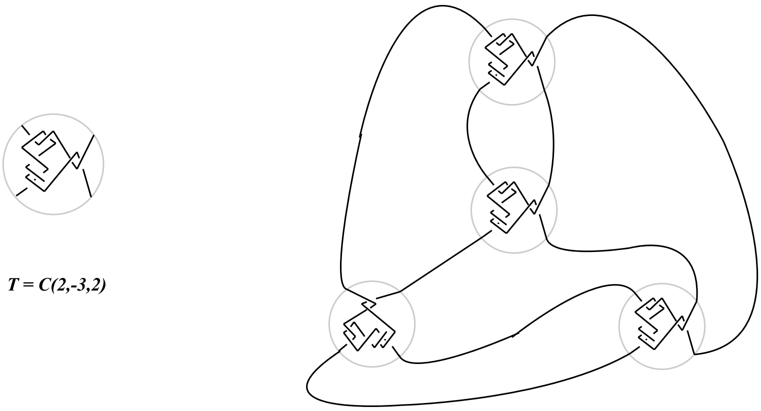


Fig. 6. D: figure-8 knot, $T = C(2, -3, 2)$.

same way, one can obtain the next level of the skein resolution tree of $D \otimes T$ by applying skein resolution tree to T whose location is the crossing c' of the next level diagram D'. By using this step in exactly the same order of the skein resolution tree of D, one can get the skein resolution tree of $D \otimes T$.

Notice that the branches of two skein trees are exactly the same, but the values assigned at each step are different; one is A and B and the other $f(A)$ and $g(A)$. Hence if $< D; s >= A^{a(s)}B^{b(s)}d^{|s|-1}$ for a branch s of the skein tree in D, the value $< D \otimes T; s >$ for the corresponding branch of the skein tree in $D \otimes T$ is $< D \otimes T; s >= f(A)^{a(s)}g(A)^{b(s)}d^{|s|-1}$. Hence

$$< D \otimes T >= \sum_{s:\text{branch}} < D \otimes T; s >= \sum_{s:\text{branch}} f(A)^{a(s)}g(A)^{b(s)}d^{|s|-1}.$$

Example 3. The right diagram of Fig. 6 is the tensor product $D \otimes T$ of the figure-eight knot D and the rational tangle $T = C(2, -3, 2)$. One can see that, by the direct calculation,

$$\begin{gathered} < D > = 5A^2B^2 + 4A^3Bd + 4AB^3d + A^4d^2 + A^2B^2d^2 + B^4d^2 \quad \text{and} \\ C(2,-3,2) = (\frac{1}{A} - 2A^3 + 2A^7 - 2A^{11} + A^{15})U_1 + (\frac{1}{A^3} - 2A + A^5 - A^9)U_2. \end{gathered}$$

Hence, by the previous theorem, the Kauffman bracket polynomial $< D \otimes T >$ of $D \otimes T$ is

$$\begin{aligned} &5(\frac{1}{A} - 2A^3 + 2A^7 - 2A^{11} + A^{15})^2(\frac{1}{A^3} - 2A + A^5 - A^9)^2 \\ &\quad + 4(\frac{1}{A} - 2A^3 + 2A^7 - 2A^{11} + A^{15})^3(\frac{1}{A^3} - 2A + A^5 - A^9)(-A^2 - A^{-2}) \\ &\quad + 4(\frac{1}{A} - 2A^3 + 2A^7 - 2A^{11} + A^{15})(\frac{1}{A^3} - 2A + A^5 - A^9)^3(-A^2 - A^{-2}) \end{aligned}$$

$$
\begin{aligned}
&+(\frac{1}{A}-2A^3+2A^7-2A^{11}+A^{15})^4(-A^2-A^{-2})^2\\
&+(\frac{1}{A}-2A^3+2A^7-2A^{11}+A^{15})^2(\frac{1}{A^3}-2A+A^5-A^9)^2(-A^2-A^{-2})^2\\
&+(\frac{1}{A^3}-2A+A^5-A^9)^4(-A^2-A^{-2})^2\\
=&\frac{1}{A^{16}}-\frac{9}{A^{12}}+\frac{37}{A^8}-\frac{99}{A^4}+207-365A^4+557A^8-754A^{12}+916A^{16}-1006A^{20}\\
&+1006A^{24}-913A^{28}+755A^{32}-563A^{36}+380A^{40}-228A^{44}+121A^{48}-55A^{52}\\
&+21A^{56}-6A^{60}+A^{64}
\end{aligned}
$$

3. Application: the adequacy of $D \otimes T$

In 1987, L.H. Kauffamn [2] showed that the *breadth* $\beta(< L >)$ of the Kauffman bracket polynomial $< L >$, which is the difference between the maximal degree and the minimal degree of $< L >$, is bounded from above by $4c(D)$ for any diagram D of L with $c(D)$ crossings. Furthermore, if D is reduced and alternating, the equality holds. In fact, $\max\deg < L >\leq c(D)+2(|s_A|-1)$ and $\min\deg < L >\geq -c(D)-2(|s_A|-1)$ and hence $\beta(< L >)\leq 2c(D)+2(|s_A|+|s_B|-2)$, where $s_A = s_A(D)$ and $s_B = s_B(D)$ are the extreme states which assign A and B to all crossings of D, respectively.

In 1988, W.B.R. Lickorish and M.B. Thistlethwaite [3] introduced $+$*adequacy* and $-$*adequacy* of link diagrams, which are sufficient conditions for equality in the above inequalities. A diagram D is said to be $+$*adequate* if $|s_A| > |s|$ for any state s whose only one crossing is assigned with a B-split. Similarly, D is said to be $-$*adequate* if $|s_B| > |s|$ for any state s whose only one crossing is assigned with a A-split. A diagram D is said to be *adequate* if it is both +adequate and −adequate. A link L is said to be *adequate* if it admits an adequate diagram. Any reduced and alternating diagram of a link is adequate. Note that if D is +adequate, $\max\deg < D >= c(D)+2(|s_A|-1)$, and if D is −adequate, $\min\deg < D >= -c(D)-2(|s_A|-1)$, and hence if D is adequate, then $\beta(< D >) = 2c(D)+2(|s_A|+|s_B|-2)$. By using these results, M.B. Thistlethwaite [5] showed that every adequate diagram is a minimal diagram. The adequacy of a diagram can be generalized for a tangle similarly.

Definition 4. A diagram T of a tangle is said to be $+$*adequate* if $|s_A| > |s|$ for any state s whose only one crossing is assigned with a B-split. Similarly, T is said to be $-$*adequate* if $|s_B| > |s|$ for any state s whose only one crossing is assigned with a A-split. T is said to be *adequate* if it is both +adequate and −adequate. A 4-tangle is said to be *adequate* if it admits an adequate diagram. Here $|s|$ denote sum of the number of loops and the number of arcs in the resulting tangle diagram obtained from T by splicing all crossings of D according to the state s.

Theorem 5. *Let D be a diagram of a link and T a diagram of a 4-tangle.*

1. *If D is +adequate and if T is an +adequate, then $D \otimes T$ is also +adequate.*
2. *If D is −adequate and if T is an −adequate, then $D \otimes T$ is also −adequate.*
3. *If D is adequate and if T is an adequate, then $D \otimes T$ is also adequate.*

Proof. (1) Since the extreme state $s_A(D)$ of D is obtained from D by applying A-splice at all crossings, $s_A(D)$ is exactly the same with the diagram $D \otimes U_1$. Similarly, $s_B(D)$ is exactly the same with the diagram $D \otimes U_2$. Here U_1 and U_2 are the generators of $E_4(A, B, d)$ in Fig. 4. Since $s_A(T)$ and $s_B(T)$ are 4-tangles with no crossings, they are either of the form U_1 with a number of loops or of the form U_2 with a number of loops, see Fig. 7.

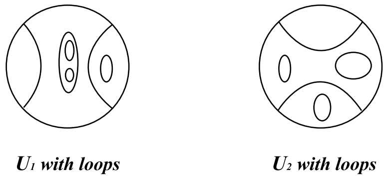


Fig. 7. Shapes of $s_A(T)$ and $s_B(T)$.

Note that, by applying A-splices to all crossings of $D \otimes T$, each tangle of $D \otimes T$ is changed into $s_A(T)$. Since $D \otimes T$ has $c(D)$ tangles, $s_A(D \otimes T)$ is $D \otimes U_1$ with $kc(D)$ loops if $s_A(T)$ consists of U_1 and k loops. Similarly, $s_B(D \otimes T)$ is $D \otimes U_2$ with $k'c(D)$ loops if $s_B(T)$ consists of U_2 and k' loops.

To check the +adequacy of $D \otimes T$, it suffice to show that $|s_A(D \otimes T)| > |s|$, where s is any state whose only one crossing, say c, is assigned with B-splice.

Since all crossings of $D \otimes T$ are located in tangles, there is a tangle T of $D \otimes T$ on which c is lying. In the state $s(T)$ of T, let l_1 and l_2 denote the two components of $s(T)$ on which c is appeared. If l_1 and l_2 are both loops, then they are different components of $s(T)$. For, if l_1 and l_2 are the same components, then by applying A-splice at c, l_1 and l_2 are appeared in the same component. Since the resulting state obtained by A-splice at c is the extreme state $s_A(T)$, $|s(T)| = |s_A(T)| + 1$, because $s(T)$ is the state of T whose the only one crossing with B-splice is c. This is a contradiction that T is +adequate. Indeed, $|s(T)| = |s_A(T)| - 1$. Since $s_A(D \otimes T)$ and $s(D \otimes T)$ are the same except l_1 and l_2, we have $|s(D \otimes T)| = |s_A(D \otimes T)| - 1$.

If one of l_1 and l_2, say l_1, is a loop and the other is an arc, by applying A-splice at c, l_1 and l_2 are resulted in one arc, and hence $|s(D \otimes T)| = |s_A(D \otimes T)| - 1$. Finally, suppose that both of l_1 and l_2 are arcs. If all loop components of $s(D \otimes T)$ and $s_A(D \otimes T)$, then the resulting states coincide with $s(D)$ and $s_A(D)$, respectively. Since D is +adequate so that $|s(D)| = |s_A(D)| - 1$, we have $|s(D \otimes T)| = |s_A(D \otimes T)| - 1$.

The proof of the second is similar with the first and the last result comes from the others.

Corollary 6. *Suppose that D is +adequate.*

1. *If D is +adequate and if* $\max\deg f(A) \geq \max\deg g(A)$*, then* $\max\deg < D \otimes T >= n \max\deg f(A) + 2\,|s_A| - 2$.
2. *If D is +adequate and if* $\max\deg f(A) \leq \max\deg g(A)$*, then* $\max\deg < D \otimes T >= n \max\deg g(A) + 2\,|s_A| - 2$.
3. *If D is −adequate and if* $\min\deg f(A) \leq \min\deg g(A)$*, then* $\min\deg < D \otimes T >= n \min\deg g(A) - 2\,|s_B| + 2$.
4. *If D is −adequate and if* $\min\deg f(A) \leq \min\deg g(A)$*, then* $\min\deg < D \otimes T >= n \min\deg f(A) - 2\,|s_B| + 2$.

Proof. We can get the results from the definition of adequacies and the equation;

$$< D \otimes T >= \sum_{s:\text{branch}} f(A)^{a(s)} g(A)^{b(s)} d^{|s|-1}.$$

4. Homfly polynomial of a link diagram with local symmetry

The *Homfly polynomial* $P_L(v, z)$ of an oriented link L is defined by the following three axioms [1,4]:

(i) $P_L(v, z)$ is invariant under ambient isotopy of L.
(ii) If L is the trivial knot, then $P_L(v, z) = 1$.
(iii) The skein relation for Homfly polynomial.

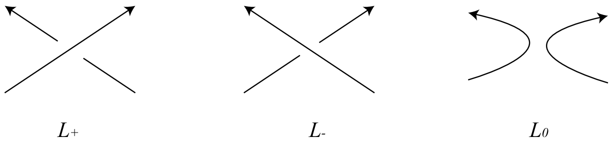


Fig. 8. The skein relation.

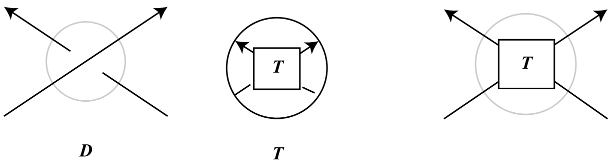


Fig. 9. Tensor product $D \otimes T$.

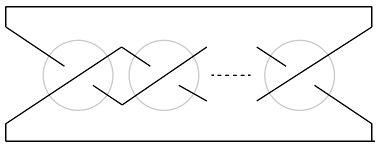


Fig. 10. n-twist.

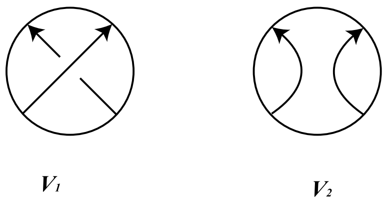


Fig. 11. Basic 4-tangles for the oriented case.

$$v^{-1}P(L_+) - vP(L_-) = zP(L_0)$$

where L_+, L_- and L_0 are the oriented link diagrams in Fig. 8, respectively.

Note that $P(L_1 \sqcup L_2) = \delta P(L_1)P(L_2)$, where $\delta = \frac{v^{-1}-v}{z}$.

If D is a positive link diagram and if T is an oriented 4-tangle, then, by replacing each crossing with T so that the orientation of D and T are compatible, one can define the *tensor product* $D \otimes T$ *of* D *and* T as in the Fig. 9.

In particular, if T_n is the diagram of n-twist in Fig. 10, $T_n \otimes T$ is a periodic link whose factor link is $T_1 \otimes T$.

Notice that any oriented 4-tangle T can be presented as a linear combination of the two basic 4-tangles V_1 and V_2 in Fig. 11 by applying suitable skein relation if needed.

Theorem 7. *Let $T_n \otimes T$ denote the periodic link described above. If $T = f(v,z)V_1 + g(v,z)V_2$, then*

$$P_{T_n \otimes T} = \sum_{i=0}^{n} f(v,z)^{n-i} g(v,z)^i \sum_{D_i} P_{D_i},$$

where D_i runs all the possible diagrams obtained from T_n by splicing i crossings.

Proof. By using the equation $T = f(v,z)V_1 + g(v,z)V_2$, one can replace each tangle of $T_n \otimes T$ with either $f(v,z)V_1$ or $g(v,z)V_2$.

Corollary 8. *Let $T_n \otimes T$ denote the periodic link described above. Suppose that $T = f(v,z)V_1 + g(v,z)V_2$. If $n = p^r$ for a prime number p and $r \geq 1$, then*

$$P_{T_n \otimes T} = f(v,z)^n P_{T_n} + \frac{v^{-1} - v}{z} g(v,z)^n \pmod{p}.$$

Proof. Suppose that $n = p$ is a prime. By the above theorem,

$$P_{T_n \otimes T} = \sum_{i=0}^{n} f(v,z)^{n-i} g(v,z)^i \sum_{D_i} P_{D_i},$$

where D_i runs all the possible diagrams obtained from T_n by splicing i crossings. Since n is prime, all diagrams obtained from T_n by splicing i crossings are ambient isotopic. Hence

$$\begin{aligned} P_{T_n \otimes T} &= \sum_{i=0}^{n} \binom{n}{i} f(v,z)^{n-i} g(v,z)^i P_{D_i} \\ &= f(v,z)^n P_{D_0} + g(v,z)^n P_{D_n} \pmod{p} \\ &= f(v,z)^n P_{T_n} + \frac{v^{-1} - v}{z} g(v,z)^n \pmod{p}. \end{aligned}$$

The last equality comes from the fact that $D_0 = T_n$ and D_n is the trivial link with two components.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A4A01009028).

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